

# ON THE COMPACTNESS PROPERTY OF EXTENSIONS OF FIRST-ORDER GÖDEL LOGIC

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**ABSTRACT.** We study three kinds of compactness in some variants of Gödel logic: compactness, entailment compactness, and approximate entailment compactness. For countable first-order underlying language we use the Henkin construction to prove the compactness property of extensions of first-order Gödel logic enriched by nullary connective or the Baaz's projection connective. In the case of uncountable first-order language we use the ultraproduct method to derive the compactness theorem.

*Keywords:* Mathematical fuzzy logic; Gödel logic; Rational Gödel logic; Compactness theorem

## 1. INTRODUCTION

Compactness theorem is one of the most important theorems in classical first-order logic. This theorem says that any finitely satisfiable theory is satisfiable. Certainly, this property provides a procedure to find models of a theory whose finite subsets have models. So, it could be considered as a foundation for model theoretical studies of any logic. Due to the fact that the model theory of mathematical fuzzy logic is still underdeveloped, study of compactness property would be a topic of interest in the area of mathematical fuzzy logic. In the case of t-norm based fuzzy logics and their extensions, this is done by several authors [Háj98, BZ98, NB01, Ger01, CN04, BYU10, Pre03, Cin05, TPD11, PT12].

Among t-norm based fuzzy logics, three of them are quite important (Gödel, Łukasiewicz, and product logic). So, almost all studies around compactness property are done for these triple. Note that various kinds of compactness are available for t-norm based fuzzy logic, e.g., compactness [NB01, Ger01, CN04, Cin05], entailment compactness [BZ98, Pre03, Cin05] and  $K$ -compactness [Cin05, TPD11, PT12] where  $K$  is a closed subset of standard truth value set  $[0, 1]$ . The usual compactness is the same as  $\{1\}$ -compactness. Let us remind that a logic enjoys the entailment compactness if for every theory  $T$  and sentence  $\varphi$ ,  $T \models \varphi$  implies the existence of a finite subset  $T'$  of  $T$  such that  $T' \models \varphi$ .

In first-order Gödel logic, different truth value sets cause different results about compactness. A truth value set in general is taken to be any linearly ordered Heyting algebra  $\mathbb{D}$ . The standard truth value set is commonly assumed to be a Gödel set which is a closed subset of  $[0, 1]$  containing 0 and 1. The first-order Gödel logic whose truth value set is a Gödel set  $V$  is denoted by  $\mathfrak{G}_V$ . Recently, all three mentioned instance of compactness are studied for Gödel set  $\mathfrak{G}_V$  [PT12, Pre03]. Furthermore, [Pre03] studies the extensions of Gödel logic  $\mathfrak{G}_V$  by  $\Delta$  Baaz projection connective.

In Łukasiewicz logic as well as its extension such as rational Pavelka logic (RPL) and continuous first-order logic (CFO) the compactness theorem is extensively studied in several frameworks [Háj98, CEG07, EGGN07, Pav79, TPD11, BYU10]. The continuity of logical connectives of Łukasiewicz logic with respect to the usual order topology on  $[0, 1]$  is the main reason for the compactness theorem to be held in these logics. By different methods such as Henkin construction, Pavelka completeness, and ultraproduct method the compactness theorem proved in these logics.

Study of the compactness property for extensions of Gödel logic is different from two viewpoints. Firstly, the Gödel logic implication is not a continuous function with respect to the usual order topology on Gödel sets. So, the Pavelka method and ultraproduct method could not be used directly in extensions

of Gödel logic. However, a modification of these methods may work here. Secondly, the corresponding algebras with respect to the extensions of Gödel logics can not be embedded into the standard truth value sets (Gödel sets) unless the algebras are at most countable. But, we need such an embedding to prove the compactness theorem by the Henkin construction. So, the Henkin construction only works for theories with at most countable first-order underlying languages.

We consider two approaches to prove the compactness property in extensions of Gödel logics. The first one is based on the Henkin construction, and so it works only for theories with at most countable first-order underlying languages. The other approach is based on the ultraproduct method. We consider a metric on Gödel sets such that the logical connectives of the corresponding extension of Gödel logic are continuous with respect to the new metric.

In Łukasiewicz logic if " $e$ " is a similarity relation, then the interpretation of " $1 - e$ " becomes a pseudometric. But, we have not a logical connective such as "minus" in Gödel logic. However, if one considers a reverse semantical meaning on truth value set, the interpretation of similarity relation will be a pseudometric in any t-norm based fuzzy logic. Furthermore, assuming such a semantic leads to obtain a pseudometric on the corresponding algebras of the logic. Besides these two pseudometrics, the continuity of logical connectives and also continuity of the interpretation of function and predicate symbols are directly intelligible. So, using the ultraproduct method motivates us to consider a reverse semantical meaning on Gödel sets which we call it the metrically semantic of the logic. Thus, 0 stands for absolute truth while 1 for absolute falsity. Anyway, we present a translation of results for the everyday Gödel logic in the final section.

This paper is organized as follows. In the next section, we introduce the main notions of extensions of Gödel logic such as logical connectives, metrically semantic, satisfiability, and so forth. Section 3 studies the main concept of the paper by studying different notions of compactness in several kinds of extensions of Gödel logic. Section 4 presents the notion of ultrametric structure and prove the compactness property for some variants of Gödel logics without any limitation on the size of the underlying first-order language. In the last section, a translation of results for the usual semantic (in which 0 stands for falsity) is given.

## 2. PRELIMINARIES

The logic that we will consider in this paper is the Gödel logic whose semantic is based on Gödel sets, i.e, subsets of unite interval  $[0, 1]$  containing 0 and 1 and closed under the standard order topology. Logical symbols of the first-order Gödel logic are the usual connectives of classical first-order logic  $\{\wedge, \rightarrow, \perp\}$  together with the quantifiers  $\{\forall, \exists\}$  and a countable set of variables.

We use a reverse semantical meaning on the set of truth values. Indeed, this assumption makes the interpretation of similarity relation a pseudometric. So, semantically 0 is the absolute truth and 1 is the absolute falsity of the truth value set.

When a Gödel set  $V$  is considered as the set of truth values, we use the notion  $\mathfrak{G}_V$  for corresponding Gödel logic. Enriching  $\mathfrak{G}_V$  by a countable set of nullary connectives  $\bar{A} = \{\bar{r} : r \in A \subseteq V \setminus \{0, 1\}\}$  leads to an extension of Gödel logic,  $\mathfrak{G}_{V,A}$ . Observe that the nullary connective  $\bar{1}$  is actually  $\perp$ . Another extension of Gödel logic is obtained by adding the unary connective  $\Delta$ . The corresponding Gödel logics equipped by  $\Delta$  are denoted by  $\mathfrak{G}_V^\Delta$  or  $\mathfrak{G}_{V,A}^\Delta$ , respectively. Lets take an abbreviation for some Gödel logics:

- $\mathfrak{G}_{\mathbb{R}}$ :  $V = [0, 1]$  and  $A = \emptyset$ .
- $\mathfrak{G}_{\downarrow}$ :  $V = [0, 1]_{\downarrow}$  and  $A = (0, 1)_{\downarrow}$  where  $[0, 1]_{\downarrow} = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$  and  $(0, 1)_{\downarrow} = [0, 1]_{\downarrow} \setminus \{0, 1\}$ .
- $\mathfrak{G}_n$ :  $V = \{r_1, \dots, r_n\} \cup \{0, 1\}$  and  $A = \{r_1, \dots, r_n\}$  where  $0 < r_1 < r_2 < \dots < r_{n-1} < r_n < 1$ .
- $\mathfrak{G}_{\downarrow}^*$ :  $V = [0, 1]$  and  $A = (0, 1)_{\downarrow}$ .
- $\mathfrak{G}_n^*$ :  $V = [0, 1]$  and  $A = \{r_1, \dots, r_n\}$  where  $0 < r_1 < r_2 < \dots < r_{n-1} < r_n < 1$ .
- $RGL$ :  $V = [0, 1]$  and  $A = (0, 1) \cap \mathbb{Q}$ .

Within this paper, we assume that  $\mathcal{L}$  is a first-order language.  $\mathcal{L}$ -terms and  $\mathcal{L}$ -formulas are constructed as in classical first-order logic. Basic notions of free and bound variable,  $\mathcal{L}$ -sentence and  $\mathcal{L}$ -theory are defined as usual. In particular, note that  $\bar{r}$  is an  $\mathcal{L}$ -sentence in  $\mathfrak{G}_{V,A}$  for each  $r \in A$ . The set of  $\mathcal{L}$ -formulas and  $\mathcal{L}$ -sentences are denoted by  $Form(\mathcal{L})$  and  $Sent(\mathcal{L})$ , respectively. When there is no danger of confusion, we may omit the prefix  $\mathcal{L}$  and simply write a term, formula, etc.

**Definition 2.1.** For a given language  $\mathcal{L}$ , an  $\mathcal{L}$ -structure  $\mathcal{M}$  in Gödel logic  $\mathfrak{G}_{V,A}$  is a nonempty set  $M$  called the universe of  $\mathcal{M}$  together with:

- (1) for any n-ary predicate symbol  $P$  of  $\mathcal{L}$ , a function  $P^{\mathcal{M}} : M^n \rightarrow V$ ,
- (2) for any n-ary function symbol  $f$  of  $\mathcal{L}$ , a function  $f^{\mathcal{M}} : M^n \rightarrow M$ ,
- (3) for any constant symbol  $c$  of  $\mathcal{L}$ , an element  $c^{\mathcal{M}}$  in the universe of  $\mathcal{M}$ .

When the underlying language is clear,  $\mathcal{M}$  is called a structure.

For each  $\alpha \in \mathcal{L}$ ,  $\alpha^{\mathcal{M}}$  is called the *interpretation* of  $\alpha$  in  $\mathcal{M}$ . The interpretation of terms is defined as follows.

**Definition 2.2.** For every n-tuple variable  $\bar{x}$  and every term  $t(\bar{x})$ , the interpretation of  $t(\bar{x})$  in  $\mathcal{M}$  is a function  $t^{\mathcal{M}} : M^n \rightarrow M$  such that

- (1) if  $t(\bar{x}) = x_i$  for  $1 \leq i \leq n$ , then  $t^{\mathcal{M}}(\bar{a}) = a_i$ ,
- (2) if  $t(\bar{x}) = c$  then  $t^{\mathcal{M}}(\bar{a}) = c^{\mathcal{M}}$ ,
- (3) if  $t(\bar{x}) = f(t_1(\bar{x}), \dots, t_n(\bar{x}))$  then  $t^{\mathcal{M}}(\bar{a}) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \dots, t_n^{\mathcal{M}}(\bar{a}))$ .

Considering 0 as the absolute truth makes some changes in some semantical issues. For example, the interpretation of  $\varphi \wedge \psi$  in a structure is absolutely true whenever the interpretation of both of them are absolutely true, i.e, the maximum of their interpretations must be absolutely true. The interpretation of formulas is defined as follows.

**Definition 2.3.** The interpretation of a formula  $\varphi(\bar{x})$  in an  $\mathcal{L}$ -structure  $\mathcal{M}$  in the Gödel logic  $\mathfrak{G}_{V,A}$  ( $\mathfrak{G}_{V,A}^{\Delta}$ ) is a function  $\varphi^{\mathcal{M}} : M^n \rightarrow V$  which is inductively determined as follows.

- (1)  $\perp^{\mathcal{M}} = 1$ , and for each  $r \in A \cup \{0\}$ ,  $\bar{r}^{\mathcal{M}} = r$ .
- (2) For every n-ary predicate symbol  $P$ ,  $P^{\mathcal{M}}(t_1(\bar{a}), \dots, t_n(\bar{a})) = P^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \dots, t_n^{\mathcal{M}}(\bar{a}))$ .
- (3)  $(\varphi \wedge \psi)^{\mathcal{M}}(\bar{a}) = \max\{\varphi^{\mathcal{M}}(\bar{a}), \psi^{\mathcal{M}}(\bar{a})\}$ .
- (4)  $(\varphi \rightarrow \psi)^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{M}}(\bar{a}) \div \psi^{\mathcal{M}}(\bar{a})$ , where  $x \div y = \begin{cases} 0 & x \geq y, \\ y & x < y. \end{cases}$
- (5) If  $\varphi(\bar{x}) = \forall y \psi(y, \bar{x})$  then  $\varphi^{\mathcal{M}}(\bar{a}) = \sup_{b \in M} \{\psi^{\mathcal{M}}(b, \bar{a})\}$ .
- (6) If  $\varphi(\bar{x}) = \exists y \psi(y, \bar{x})$  then  $\varphi^{\mathcal{M}}(\bar{a}) = \inf_{b \in M} \{\psi^{\mathcal{M}}(b, \bar{a})\}$ .
- (7) (Only for  $\mathfrak{G}_{V,A}^{\Delta}$ )  $(\Delta(\varphi))^{\mathcal{M}}(\bar{a}) = \begin{cases} 0 & \varphi^{\mathcal{M}}(\bar{a}) = 0, \\ 1 & \text{otherwise.} \end{cases}$

Observe that since  $V$  is a closed subset of  $[0, 1]$ , all infima and suprema exist. One can consider an abbreviation for compound connectives  $\neg, \vee, \Rightarrow$  and  $\leftrightarrow$ .

- $\neg\varphi := \varphi \rightarrow \perp$ .
- $\varphi \vee \psi := ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$ .
- $\varphi \Rightarrow \psi := (\psi \rightarrow \varphi) \rightarrow \psi$ .
- $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ .

The interpretations of formulas including these new connectives can be computed as follows.

- $\neg\varphi^{\mathcal{M}}(\bar{a}) = \begin{cases} 0 & \varphi^{\mathcal{M}}(\bar{a}) = 1, \\ 1 & \varphi^{\mathcal{M}}(\bar{a}) < 1. \end{cases}$
- $(\varphi \vee \psi)^{\mathcal{M}}(\bar{a}) = \min\{\varphi^{\mathcal{M}}(\bar{a}), \psi^{\mathcal{M}}(\bar{a})\}$ .

- $(\varphi \Rightarrow \psi)^{\mathcal{M}}(\bar{a}) = \begin{cases} 0 & \varphi^{\mathcal{M}}(\bar{a}) > \psi^{\mathcal{M}}(\bar{a}) > 0, \\ \psi^{\mathcal{M}}(\bar{a}) & \varphi^{\mathcal{M}}(\bar{a}) \leq \psi^{\mathcal{M}}(\bar{a}). \end{cases}$
- $(\varphi \leftrightarrow \psi)^{\mathcal{M}}(\bar{a}) = d_{\max}(\varphi^{\mathcal{M}}(\bar{a}), \psi^{\mathcal{M}}(\bar{a}))$ , where  $d_{\max}(x, y) = \begin{cases} 0 & x = y, \\ \max\{x, y\} & x \neq y. \end{cases}$

**Definition 2.4.** Let  $\varphi(\bar{x})$  be an  $\mathcal{L}$ -formula and  $T$  be an  $\mathcal{L}$ -theory.

- (1) An  $\mathcal{L}$ -structure  $\mathcal{M}$  is called a model of  $\varphi(\bar{x})$ , if there is  $\bar{a} \in M^n$  such that  $\varphi^{\mathcal{M}}(\bar{a}) = 0$ . In such a case, we write  $\mathcal{M} \models \varphi(\bar{a})$ .
- (2)  $\varphi(\bar{x})$  is called a satisfiable formula if there is an  $\mathcal{L}$ -structure  $\mathcal{M}$  which models  $\varphi(\bar{x})$ .
- (3) If an  $\mathcal{L}$ -structure  $\mathcal{M}$  models all sentences of  $T$ , we call  $T$  a *satisfiable* theory and write  $\mathcal{M} \models T$ .
- (4)  $T$  is called finitely satisfiable if every finite subset of  $T$  has a model.
- (5) For an  $\mathcal{L}$ -sentence  $\varphi$  we say that  $T$  entails  $\varphi$ ,  $T \models \varphi$ , if every model of  $T$  models  $\varphi$ . We write  $T \models^f \varphi$  if there exists a finite subset  $S$  of  $T$  so that  $S \models \varphi$ . If there is no finite subset  $S$  of  $T$  such that  $S \models \varphi$ , we write  $T \not\models^f \varphi$ . We use  $\models \varphi$  instead of  $\emptyset \models \varphi$ .

For any Gödel set  $V$  and  $A \subseteq V$ , the axioms of the Gödel logic  $\mathfrak{G}_{V,A}$  are the axioms of first-order Gödel logic [Háj98] together with the book-keeping axioms listed in Table 1.

TABLE 1. Axioms and Rules of  $\mathfrak{G}_{V,A}$

Axioms of first-order Gödel logic		
(G1)	$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$	
(G2)	$(\varphi \wedge \psi) \rightarrow \varphi$	
(G3)	$(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$	
(G4)	$\varphi \rightarrow (\varphi \wedge \varphi)$	
(G5)	$(\varphi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow ((\varphi \wedge \psi) \rightarrow \chi)$	
(G6)	$((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$	
(G7)	$\bar{1} \rightarrow \varphi$	
(GV1)	$(\forall x \varphi(x)) \rightarrow \varphi(t)$	( $t$ substitutable for $x$ in $\varphi(x)$ )
(GV2)	$(\forall x (\psi \rightarrow \varphi(x))) \rightarrow (\psi \rightarrow (\forall x \varphi(x)))$	( $x$ not free in $\psi$ )
(GV3)	$(\forall x (\psi \vee \varphi(x))) \rightarrow (\psi \vee (\forall x \varphi(x)))$	( $x$ not free in $\psi$ )
(G∃1)	$\varphi(t) \rightarrow (\exists x \varphi(x))$	( $t$ substitutable for $x$ in $\varphi(x)$ )
(G∃2)	$(\forall x (\varphi(x) \rightarrow \psi)) \rightarrow ((\exists x \varphi(x)) \rightarrow \psi)$	( $x$ not free in $\psi$ )
Book-keeping axioms for nullary connectives		
(RG1)	$\bar{r} \wedge \bar{s} \leftrightarrow \overline{\max\{r, s\}}$	
(RG2(a))	$\bar{r} \rightarrow \bar{s}$	(for $r \geq s$ )
(RG2(b))	$(\bar{r} \rightarrow \bar{s}) \leftrightarrow \bar{s}$	(for $r < s$ )
(RG3)	$\neg\neg\bar{r}$	(for $r < 1$ )
Rules		
(Mp)	$\varphi, (\varphi \rightarrow \psi) \vdash \psi$	
(Gen)	$\varphi \vdash \forall x \varphi$	

**Definition 2.5.** An  $\mathcal{L}$ -sentence  $\varphi$  is proved by an  $\mathcal{L}$ -theory  $T$ ,  $T \vdash \varphi$ , whenever there is a finite sequence  $\{\varphi_i\}_{i=1}^n$  of  $\mathcal{L}$ -sentences such that:

- for each  $1 \leq i \leq n$  either  $\varphi_i \in T$  or  $\varphi_i$  is an axiom or it is followed by rules from axioms and other  $\varphi_j$ 's for  $1 \leq j < i$ .
- $\varphi_n = \varphi$ .

We write  $\vdash \varphi$  whenever  $\emptyset \vdash \varphi$ .  $T$  is called a consistent theory if  $T \not\vdash \perp$ .

Note that if  $A \neq \emptyset$  then for any  $0 < r < 1$ ,  $T = \{\bar{r}\}$  is a consistent theory in  $\mathfrak{G}_{V,A}$ . However,  $T$  is not a satisfiable theory. In the next section we introduced the notion of strongly consistency which is equivalent to the notion of satisfiability in some extensions of Gödel logics. The deduction theorem follows easily.

**Theorem 2.6.** *In the Gödel logic  $\mathfrak{G}_{V,A}$ , for an  $\mathcal{L}$ -theory  $T$  and  $\mathcal{L}$ -sentences  $\varphi$  and  $\psi$ ,*

$$T \cup \{\varphi\} \vdash \psi \text{ if and only if } T \vdash \varphi \rightarrow \psi.$$

Obviously if  $T \vdash \varphi$ , then  $T \models \varphi$  and also  $T \models^f \varphi$ . In spite of first-order logic, the concept of proof does not coincide completely with the concept of finitely entailment in Gödel logics enriched by nullary connectives.

**Example 2.7.** Let  $A \neq \emptyset$ . One could easily verify that in the Gödel logic  $\mathfrak{G}_{[0,1],A}$  if  $\mathcal{L} = \{\rho\}$  where  $\rho$  is a nullary predicate symbol and  $r \in A \setminus \{0, 1\}$ , then  $\neg \neg \rho \rightarrow \bar{r} \models \neg \rho$  while  $\neg \neg \rho \rightarrow \bar{r} \not\models \neg \rho$ .

*Remark 2.8.* For Gödel logics enriched by  $\Delta$  connective, there are some additional axioms and rules.

- $\Delta 1)$   $\Delta \varphi \vee \neg \Delta \varphi$ .
- $\Delta 2)$   $\Delta(\varphi \vee \psi) \rightarrow (\Delta \varphi \vee \Delta \psi)$ .
- $\Delta 3)$   $\Delta \varphi \rightarrow \varphi$ .
- $\Delta 4)$   $\Delta \varphi \rightarrow \Delta \Delta \varphi$ .
- $\Delta 5)$   $\Delta(\varphi \rightarrow \psi) \rightarrow (\Delta \varphi \rightarrow \Delta \psi)$ .
- $\Delta R)$   $\varphi \vdash \Delta \varphi$ .

**Definition 2.9.** A Gödel logic has complete recursive axiomatization whenever  $\models \varphi$  if and only if  $\vdash \varphi$  for every  $\mathcal{L}$ -sentence  $\varphi$ . This kind of completeness is sometimes called *weak completeness*.

**Definition 2.10.** A Gödel logic is said to have the strong completeness whenever for every  $\mathcal{L}$ -theory  $T$  and  $\mathcal{L}$ -sentence  $\varphi$ ,  $T \models \varphi$  if and only if  $T \vdash \varphi$ .

First-order Gödel logic  $\mathfrak{G}_{\mathbb{R}}$  admits both kinds of completeness with respect to any countable first-order language [Háj98]. When  $A \neq \emptyset$  example 2.7 shows that the strong completeness fails in  $\mathfrak{G}_{[0,1],A}$  while it is shown that  $\mathfrak{G}_{[0,1],A}$  is completely recursive axiomatizable [EGN09].

One of the most useful tools in model theory of classical first-order logic is the compactness theorem. In the case of mathematical fuzzy logic this theorem has different aspects.

**Definition 2.11.** A Gödel logic is said to enjoy the *entailment compactness* whenever for any theory  $T$  and sentence  $\varphi$ ,

$$T \models \varphi \text{ if and only if } T \models^f \varphi.$$

**Definition 2.12.** The Gödel logic  $\mathfrak{G}_{V,A}$  ( $\mathfrak{G}_{V,A}^{\Delta}$ ) has the *approximate entailment compactness* property if for every theory  $T$  and sentence  $\varphi$ ,

$$T \models \varphi \text{ if and only if } T \models \bar{r} \rightarrow \varphi \text{ for all } r \in A \cup \{1\}.$$

**Definition 2.13.** We say that a Gödel logic has the *compactness* property if for every theory  $T$ ,  $T$  is satisfiable if and only if  $T$  is finitely satisfiable.

Since a proof is a finite sequence of conclusions, we have the following theorem.

**Theorem 2.14.** *If a logic admits the strong completeness then it enjoys the triple kinds of compactness mentioned above.*

Specially, in the Gödel logic  $\mathfrak{G}_{\mathbb{R}}$  both entailment compactness and compactness hold.

**Theorem 2.15.** [Pre03] *The entailment compactness and complete recursive axiomatization (weak completeness) are equivalent in Gödel logic  $\mathfrak{G}_V$ .*

Furthermore, Prening [Pre03] shows the Gödel logic  $\mathfrak{G}_V$  admits the entailment compactness property if and only if either  $V$  is a finite Gödel set or the perfect kernel of  $V$  includes 1 or the perfect kernel of  $V$  is nonempty and 1 is an isolated point of  $V$ . Particularly, he shows that the entailment compactness fails in  $\mathfrak{G}_V$  for countable Gödel set  $V$ .

Later, Pourmahdian et al. [PT12] show that if  $V$  is a finite Gödel set or the perfect kernel of  $V$  includes 1 or 1 is an isolated point of  $V$  then  $\mathfrak{G}_V$  admits the compactness property.

### 3. COMPACTNESS IN GÖDEL LOGIC $\mathfrak{G}_{[0,1],A}$

In this section, we study the compactness property of Gödel logics  $\mathfrak{G}_{[0,1],A}$  and also  $\mathfrak{G}_{[0,1],A}^{\Delta}$ . From now on assume that  $A'$  is denoted for the set of limit points of  $A$  in a Gödel set  $V$  with respect to the order topology on  $V$ . Firstly, note that if  $A$  has a limit point  $a \neq 0$  with respect to the order topology and  $a \in A \cup \{1\}$ , then the compactness fails in  $\mathfrak{G}_{[0,1],A}$  as well as  $\mathfrak{G}_{[0,1],A}^{\Delta}$ .

**Example 3.1.** Let  $a \in A' \cap (A \cup \{1\})$  and assume that  $\mathcal{L} = \{\rho\}$  where  $\rho$  is a nullary predicate symbol. Suppose that  $\{r_i\}_{i=1}^{\infty} \subseteq A \setminus \{a\}$  is an increasing (decreasing) sequence whose limit in  $V$  is  $a$ . Let  $T = \{\bar{a} \Rightarrow \rho\} \cup \{\rho \rightarrow \bar{r}_i\}_{i=1}^{\infty}$  ( $T = \{\rho \Rightarrow \bar{a}\} \cup \{\bar{r}_i \rightarrow \rho\}_{i=1}^{\infty}$ ). Obviously  $T$  is finitely satisfiable, but it is not satisfiable.

**Example 3.2.** Let  $a \in A'$  but  $a \notin A \cup \{1\}$ . Also assume that there is an increasing sequence  $\{r_i\}_{i=1}^{\infty} \subseteq A$  and a decreasing sequence  $\{s_i\}_{i=1}^{\infty} \subseteq A$  so that  $\lim_i r_i = a = \lim_i s_i$ . Let  $\mathcal{L} = \{\rho, R(x)\}$  where  $\rho$  is a nullary predicate symbol and  $R(x)$  is a unary predicate symbol. Let  $T = \{\exists x ((\bar{r}_{i+1} \rightarrow R(x)) \wedge (R(x) \rightarrow \bar{r}_i))\}_{i=1}^{\infty} \cup \{\bar{s}_i \Rightarrow (\forall x R(x))\}_{i=1}^{\infty} \cup \{(\forall x R(x)) \Rightarrow \rho\} \cup \{\rho \rightarrow \bar{r}_i\}_{i=1}^{\infty}$ .  $T$  is finitely satisfiable, but it is not satisfiable. Indeed if  $\mathcal{M} \models T$ , then  $(\forall x R(x))^{\mathcal{M}} = a$  and so the interpretation of  $\rho$  in  $\mathcal{M}$  makes no sense.

Specially,  $RGL$  does not admit the compactness property. However, if one consider some non-standard truth value set, the compactness may hold on  $RGL$ . [KPT13] prove that the compactness property is hold on  $RGL$  within a semantic on the non-standard truth value set  $I = [0, 1]^2 \setminus \{(0, r) : r > 0\}$ .

Now, using the Henkin construction, we show in the case that the set of limit points of  $A$  is at most  $\{0\}$ , the Gödel logic  $\mathfrak{G}_{[0,1],A}$  admits the compactness property. Observe that this method is based on constructing the Gödel algebra of equivalence classes of formulas modulo a theory, and then embedding this Gödel algebra into the unit interval  $[0, 1]$ , where the countability of the language  $\mathcal{L}$  is a prerequisite necessary assumption for existence of such an embedding. In the next section, we prove the compactness property for some extensions of Gödel logics in which the requirement of such an assumption is not obligatory.

**Definition 3.3.** The Gödel algebra with respect to the Gödel logic  $\mathfrak{G}_{V,A}$  is a bounded lattice  $\mathbb{D} = \langle D, \wedge, \vee, 0^{\mathbb{D}}, 1^{\mathbb{D}} \rangle$  together with a binary operation  $\rightarrow$  and for each  $r \in A \setminus \{0, 1\}$  an element  $r^{\mathbb{D}} \in D$  such that:

- (1)  $\wedge$  is the join (lub) operator and  $\vee$  is the meet (glb) operator.
- (2)  $\wedge$  and  $\rightarrow$  form an adjoint pair, i.e., for all  $a, b, c \in D$ ,

$$a \wedge b \geq_{\mathbb{D}} c \text{ iff } a \geq_{\mathbb{D}} b \rightarrow c,$$

where  $a \geq_{\mathbb{D}} b$  if and only if  $a \wedge b = b$ .

- (3)  $D$  is pre-linear, i.e., for all  $a, b \in D$ ,

$$(a \dot{\rightarrow} b) \vee (b \dot{\rightarrow} a) = 0^{\mathbb{D}}.$$

- (4)  $r^{\mathbb{D}} \wedge s^{\mathbb{D}} = \max\{r, s\}^{\mathbb{D}}.$
- (5)  $r^{\mathbb{D}} \dot{\rightarrow} s^{\mathbb{D}} = 0^{\mathbb{D}}$  iff  $r \geq s.$
- (6)  $r^{\mathbb{D}} \dot{\rightarrow} s^{\mathbb{D}} = s^{\mathbb{D}}$  iff  $r < s.$
- (7)  $0^{\mathbb{D}} < r^{\mathbb{D}} < 1^{\mathbb{D}}$  for all  $0 < r < 1.$

The Gödel algebra with respect to the Gödel logic  $\mathfrak{G}_{V,A}^{\Delta}$  is formed by the corresponding Gödel algebra with respect to  $\mathfrak{G}_{V,A}$ , i.e.,  $\mathbb{D} = \langle D, \wedge, \vee, \dot{\rightarrow}, \{r^{\mathbb{D}} : r \in A \cup \{0, 1\}\} \rangle$  together with a unary operation  $\delta^{\mathbb{D}}$  which acts as follows.

- (8)  $\delta^{\mathbb{D}}(0^{\mathbb{D}}) = 0^{\mathbb{D}}.$
- (9)  $\delta^{\mathbb{D}}(a) = 1^{\mathbb{D}}$  for all  $a \in \mathbb{D} \setminus \{0^{\mathbb{D}}\}.$

**Example 3.4.** The standard Gödel algebra with respect to  $\mathfrak{G}_{[0,1],A}$  is  $[0, 1]_A = \langle [0, 1], \max, \min, \dot{\rightarrow}, \{r : r \in A \cup \{0, 1\}\} \rangle$ . The standard Gödel algebra corresponding to  $\mathfrak{G}_n^*$  and  $\mathfrak{G}_{\downarrow}^*$  is denoted by  $[0, 1]_n^*$  and  $[0, 1]_{\downarrow}^*$ , respectively.

**Lemma 3.5.** *If  $A' \subseteq \{0\}$ , then any countable linear ordered Gödel algebra  $\mathbb{D}$  with respect to  $\mathfrak{G}_{[0,1],A}$  can be continuously embedded into the standard Gödel algebra  $[0, 1]_A$  (i.e. an embedding that preserves all suprema and infima that exist in  $\mathbb{D}$ ). Particular, any countable linear ordered Gödel algebra  $\mathbb{D}$  with respect to  $\mathfrak{G}_n^*$  and  $\mathfrak{G}_{\downarrow}^*$  can be continuously embedded into the standard Gödel algebra  $[0, 1]_n^*$  and  $[0, 1]_{\downarrow}^*$ , respectively.*

*Proof.* As [Háj98, Lemma 5.3.1], set  $D' = D \times \{0\} \cup \{\{u\} \times ((0, 1) \cap \mathbb{Q}) : u \text{ has no successor in } D\}$  which ordered lexicographically by induced ordering  $\leq_{\mathbb{D}}$  of  $\mathbb{D}$ . By setting  $r^{\mathbb{D}'} = (r, 0)$  for each nullary connective  $\bar{r}$ , one can easily construct a countable densely linearly ordered Gödel algebra  $\mathbb{D}'$ . Furthermore, the mapping  $u \rightarrow (u, 0)$  is a continuous embedding from  $\mathbb{D}$  into  $\mathbb{D}'$  wherein the image of  $r^{\mathbb{D}}$  is  $r^{\mathbb{D}'}$ . There are two cases.

- Case 1:  $A' = \emptyset$ . Thus,  $A$  is a finite set. So, the proof is similar to the proof for Gödel logic [Háj98, Lemma 5.3.2] with an easy adaptation of the back and forth method for embedding countable densely linearly ordered Gödel algebra  $\mathbb{D}'$  into  $[0, 1]_A$ .
- Case 2:  $A' = \{0\}$ . So, there is a decreasing sequence  $\{r_i\}_{i \in \mathbb{N}}$  in the open unit interval  $(0, 1)$  so that  $A = \{r_i\}_{i \in \mathbb{N}}$  and  $\lim_i r_i = 0$ . Let  $\|u\| = \inf\{r : u \leq_{\mathbb{D}'} r^{\mathbb{D}'}\}$  for any  $u \in D'$ . Define the equivalence relation  $\sim$  on  $D'$  by

$$u \sim v \text{ if and only if } \|u\| = \|v\|.$$

Now, we have

- $[0^{\mathbb{D}'}]_{\sim} = \{u \in D' : u \leq_{\mathbb{D}'} r_i^{\mathbb{D}'} \text{ for all } i \in \mathbb{N}\},$
- if  $\|u\| = r_i$  then  $[u]_{\sim} = \{u \in D' : r_{i+1}^{\mathbb{D}'} <_{\mathbb{D}'} u \leq_{\mathbb{D}'} r_i^{\mathbb{D}'}\},$
- $[1^{\mathbb{D}'}]_{\sim} = \{u \in D' : r_1^{\mathbb{D}'} <_{\mathbb{D}'} u \leq_{\mathbb{D}'} 1^{\mathbb{D}'}\}.$

For each  $u \in D'$  if  $\|u\| = r \in A$ , obviously  $[u]_{\sim}$  can be continuously embedded into  $(r_{i+1}, r_i]$  by means of a function  $f_r$ . Also  $[1^{\mathbb{D}'}]_{\sim}$  continuously embedded into  $(r_1, 1]$  by a function like as  $f_1$ . Let  $f_0$  be the trivial constant function from  $[0^{\mathbb{D}'}]_{\sim}$  into  $\{0\}$ . Now, the function  $f = \cup\{f_r : r \in A \cup \{0, 1\}\}$  fulfills the proof.

□

### 3.1. Usual Compactness.

As already mentioned, if  $A \neq \emptyset$  then for any  $0 < r < 1$ ,  $T = \{\bar{r}\}$  is a consistent theory in  $\mathfrak{G}_{V,A}$  which is not satisfiable. So, when  $A \neq \emptyset$  we use the "strongly consistency" instead of "consistency".

**Definition 3.6.** An  $\mathcal{L}$ -theory  $T$  is called strongly consistent if  $T \not\models \bar{r}$  for  $r \in A \cup \{1\}$  (i.e,  $r > 0$ ).



Observe that every satisfiable theory is strongly consistent. We show that when  $A' \subseteq \{0\}$ , strongly consistent theories are satisfiable. Note that Examples 3.1 and 3.2 gives strongly consistent theories which are not satisfiable in the Gödel logic  $\mathfrak{G}_{[0,1],A}$  when  $A' \not\subseteq \{0\}$ .

Two different concepts "finitely entailment" and "proof" bring us two kinds of Henkin and complete theories.

**Definition 3.7.** Let  $T$  be an  $\mathcal{L}$ -theory.

- (1)  $T$  is Henkin if for every universal  $\mathcal{L}$ -formula  $\forall x \varphi(x)$  that is not finitely entailed by  $T$ , there is a witness constant symbol  $c$  in  $\mathcal{L}$  such that  $T \not\models \varphi(c)$ .
- (2)  $T$  is deductively Henkin or d-Henkin if for every universal  $\mathcal{L}$ -formula  $\forall x \varphi(x)$  that is not proved by  $T$ , there is a witness constant symbol  $c$  in  $\mathcal{L}$  such that  $T \not\vdash \varphi(c)$ .
- (3)  $T$  is called a complete theory if for any pair of  $\mathcal{L}$ -sentences  $(\varphi, \psi)$ , either  $T \models^f \varphi \rightarrow \psi$  or  $T \models^f \psi \rightarrow \varphi$ .
- (4)  $T$  is called deductively complete or d-complete theory if for any pair of  $\mathcal{L}$ -sentences  $(\varphi, \psi)$ , either  $T \vdash \varphi \rightarrow \psi$  or  $T \vdash \psi \rightarrow \varphi$ .

The following theorem leads to deduced the compactness property for the Gödel logic  $\mathfrak{G}_{[0,1],A}$  when  $A' \subseteq \{0\}$ .

**Theorem 3.8.** Let  $\mathcal{L}$  be a countable first-order language. If  $A' \subseteq \{0\}$  then every strongly consistent d-complete d-Henkin  $\mathcal{L}$ -theory in  $\mathfrak{G}_{[0,1],A}$  is satisfiable.

*Proof.* Let  $T$  be a strongly consistent deductively complete d-Henkin  $\mathcal{L}$ -theory. Also let  $Lind(T)$  be the class of all  $T$ -provably equivalent  $\mathcal{L}$ -sentences, i.e., the equivalence classes  $[\varphi]_T$  of all  $\mathcal{L}$ -sentences  $\varphi$  modulo to the following equivalence relation.

$$\varphi \sim \psi \text{ if and only if } T \vdash \varphi \leftrightarrow \psi.$$

Define an ordering  $\lesssim$  on  $Lind(T)$  as follows

$$[\varphi]_T \lesssim [\psi]_T \text{ if and only if } T \vdash \psi \rightarrow \varphi.$$

Because  $T$  is a complete theory,  $(Lind(T), \lesssim)$  is a linearly ordered set. Now, we obtain a countable linearly ordered Gödel algebra  $\mathbb{L}_T$  from  $Lind(T)$  by setting,

$$\begin{aligned} [\varphi]_T \vee [\psi]_T &= [\varphi \vee \psi]_T, \\ [\varphi]_T \wedge [\psi]_T &= [\varphi \wedge \psi]_T, \\ [\varphi]_T \rightarrow [\psi]_T &= [\varphi \rightarrow \psi]_T, \\ r^{\mathbb{D}} &= [\bar{r}]_T \text{ for any nullary connective } \bar{r}. \end{aligned}$$

Axiom  $RG3$  together with the strongly consistency of  $T$  implies that  $[\bar{r}]_T \lesssim [\bar{s}]_T$  for each  $r < s$  in  $A$ . Thus, by Lemma 3.5 there is an embedding  $g$  from  $\mathbb{L}_T$  into the standard Gödel algebra  $[0, 1]_A$  such that  $[\bar{r}]_T$  mapped to  $r$ .

The canonical  $\mathcal{L}$ -structure  $\mathcal{M}_T$  of  $T$  is made as follows.

- a) The universe of  $\mathcal{M}_T$  is the set of all closed  $\mathcal{L}$ -terms  $CM(T)$ .
- b) For each n-ary function symbol  $f$ , define  $f^{\mathcal{M}_T} : CM(T)^n \rightarrow CM(T)$  by

$$f^{\mathcal{M}_T}(t_1, \dots, t_n) = f(t_1, \dots, t_n).$$

- c) For each n-ary predicate symbol  $P$ , define  $P^{\mathcal{M}_T} : CM(T)^n \rightarrow [0, 1]$  by

$$P^{\mathcal{M}_T}(t_1, \dots, t_n) = g([P(t_1, \dots, t_n)]_T).$$

□



For the case that  $A' = \emptyset$ , the compactness property of  $\mathfrak{G}_{[0,1],A}$  follows from Theorem 3.8 and the following theorem.

**Theorem 3.9.** *Let  $A' = \emptyset$ . Every strongly consistent  $\mathcal{L}$ -theory in  $\mathfrak{G}_{[0,1],A}$  is contained in a strongly consistent  $d$ -complete  $d$ -Henkin  $\mathcal{L}'$ -theory such that  $\mathcal{L} \subseteq \mathcal{L}'$ .*

*Proof.* Let  $T$  be a strongly consistent  $\mathcal{L}$  theory and  $\mathcal{L}'$  be the extended of  $\mathcal{L}$  with countably many new constant symbols. Enumerate all pairs of  $\mathcal{L}'$ -sentences by  $\{(\theta_i, \psi_i)\}_{i \in \mathbb{N}}$ . Also assume that  $\{\varphi_i(x)\}_{i \in \mathbb{N}}$  be the set of all  $\mathcal{L}'$ -formulas with one free variable. Now, we construct inductively sequences  $\{T_n\}_{n \in \mathbb{N}}$  of  $\mathcal{L}'$ -theories and  $\{\chi_n\}_{n \in \mathbb{N}}$  of  $\mathcal{L}'$ -sentences such that for each  $n \in \mathbb{N}$ ,  $T_n \not\vdash \chi_n$ .

stage 0: Let  $T_0 = T$  and

$$\chi_0 = \begin{cases} 1 & A = \emptyset, \\ \min_{r \in A} \{r\} & \text{otherwise.} \end{cases}$$

Obviously,  $\chi_0 > 0$  and since  $T$  is strongly consistent,  $T_0 \not\vdash \chi_0$ .

stage  $n+1=2i$ : Let  $\chi_{n+1} = \chi_n$ . Now, if  $T_n \cup \{\theta_i \rightarrow \psi_i\} \not\vdash \chi_n$  set  $T_{n+1} = T_n \cup \{\theta_i \rightarrow \psi_i\}$  and otherwise set  $T_{n+1} = T_n \cup \{\psi_i \rightarrow \theta_i\}$ . Since  $T_n \not\vdash \chi_n$ , either  $T_n \cup \{\theta_i \rightarrow \psi_i\} \not\vdash \chi_n$  or  $T_n \cup \{\psi_i \rightarrow \theta_i\} \not\vdash \chi_n$ . Thus  $T_{n+1} \not\vdash \chi_{n+1}$ .

stage  $n+1=2i+1$ : Let  $c_i$  be a constant symbol of  $\mathcal{L}'$  not occurring in  $\varphi_i(x)$  and the constructed objects until the current stage. Consider two cases.

Case 1: If  $T_n \not\vdash \chi_n \vee \varphi_i(c_i)$  let  $T_{n+1} = T_n$  and  $\chi_{n+1} = \chi_n \vee \varphi_i(c_i)$ . Since  $T_n \not\vdash \chi_n$ , clearly in this case  $T_{n+1} \not\vdash \chi_{n+1}$ .

Case 2: If  $T_n \vdash \chi_n \vee \varphi_i(c_i)$  set  $T_{n+1} = T_n \cup \{\chi_n \rightarrow \forall x \varphi_i(x)\}$  and  $\chi_{n+1} = \chi_n$ . Since  $T_n \vdash \chi_n \vee \varphi_i(c_i)$  using (Gen) and (GV3) we have  $T_n \vdash \chi_n \vee \forall x \varphi_i(x)$ . So, by definition of the connective  $\vee$  and the fact that  $T_n \not\vdash \chi_n$  we have  $T_n \cup \{\forall x \varphi_i(x) \rightarrow \chi_n\} \vdash \chi_n$ . Thus, using the proof-by-case property and the fact that  $T_n \not\vdash \chi_n$ , we have  $T_n \cup \{\chi_n \rightarrow \forall x \varphi_i(x)\} \not\vdash \chi_n$  that is  $T_{n+1} \not\vdash \chi_{n+1}$ .

Now, let  $T' = \bigcup_{n \in \mathbb{N}} T_n$ . Clearly  $T'$  is strongly consistent, since otherwise if  $T' \vdash \bar{r}$  for some  $r \in A \cup \{1\}$  then by (RG2(a))  $T' \vdash \chi_0$ . So, for some  $n \in \mathbb{N}$ ,  $T_n \vdash \chi_0$  which implies that  $T_n \vdash \chi_n$ , a contradiction.

On the other hand, clearly  $T'$  is deductively complete. Now, if  $T' \not\vdash \forall x \varphi_i(x)$  then  $T_{2i+1} \not\vdash \chi_{2i+1} \vee \varphi_i(c_i)$ , since otherwise by case 2 of stage  $n+1$  we have  $T_{2i+1} \vdash \chi_{2i+1} \vee \forall x \varphi_i(x)$  which implies that  $T_{2i+2} \vdash \forall x \varphi_i(x)$ , a contradiction. Thus,  $T_{2i+2} = T_{2i+1}$  and  $\chi_{2i+2} = \chi_{2i+1} \vee \varphi_i(c_i)$ . But then  $T' \not\vdash \varphi_i(c_i)$ , since otherwise  $T' \vdash \chi_{2i+2}$ , a contradiction. So,  $T'$  is a deductively complete  $d$ -Henkin  $\mathcal{L}'$ -theory.  $\square$

**Corollary 3.10.** *For countable first-order language  $\mathcal{L}$ , the Gödel logic  $\mathfrak{G}_n^*$  admit the compactness property.*

*Remark 3.11.* By Example 2.7 we know that the strong completeness fails in  $\mathfrak{G}_n^*$ . Indeed, when  $T \cup \{\varphi\} \subseteq \text{Sent}(\mathcal{L})$ ,  $A \neq \emptyset$ , and  $T \not\vdash \varphi$  one could not obtain a deductively complete  $d$ -Henkin extension  $T'$  of  $T$  such that  $T' \not\vdash \varphi$  in the Gödel logic  $\mathfrak{G}_{[0,1],A}$ . For example, the theory  $T = \{\neg\neg\rho \rightarrow \bar{r}\}$  in Example 2.7 could not be extend to a deductively complete theory  $T'$  such that  $T' \not\vdash \neg\rho$ .

The method used in Theorem 3.9 could not be used for the case that  $A' = \{0\}$ . To prove the compactness property of  $\mathfrak{G}_{[0,1],A}$  for the case that  $A' = \{0\}$  we use the following lemma.

**Lemma 3.12.** *Let  $T$  be a maximally strongly consistent  $\mathcal{L}$ -theory and  $\varphi$  and  $\psi$  be two arbitrary  $\mathcal{L}$ -sentences. For the Gödel logic  $\mathfrak{G}_{[0,1],A}$  we have,*

- (1)  $T$  is deductively complete,
- (2) if  $\varphi \vee \psi \in T$ , then either  $\varphi \in T$  or  $\psi \in T$ ,
- (3) if  $A' = \{0\}$  and  $\bar{r} \rightarrow \varphi \in T$  for  $r \in A \cup \{1\}$ , then  $\varphi \in T$ .

*Proof.* (1) and (2) are straightforward. For (3) we show that  $T \cup \{\varphi\}$  is strongly consistent. Suppose, to derive a contradiction, that  $T \cup \{\varphi\}$  is not strongly consistent. So, there is  $r \in A \cup \{1\}$  such that  $T \cup \{\varphi\} \vdash \bar{r}$ . Thus,  $T \vdash \varphi \rightarrow \bar{r}$ . Since  $A' = \{0\}$ , there is  $s \in A \cup \{1\}$  such that  $s < r$ . By the assumption  $\bar{s} \rightarrow \varphi \in T$ . i.e,  $T \vdash \bar{s} \rightarrow \varphi$ . Hence, by transitivity property of proof  $T \vdash \bar{s} \rightarrow \bar{r}$  and by  $RG2(b)$ ,  $T \vdash \bar{r}$ . A contradiction.  $\square$

Observe that by Zorn's lemma, any strongly consistent  $\mathcal{L}$ -theory  $T$  contained in a maximally strongly consistent  $\mathcal{L}$ -theory. The following theorem show that this maximally strongly consistent extension could be chosen in a language  $\mathcal{L}' \supseteq \mathcal{L}$  such that it is Henkin. So, in the light of Theorem 3.8 the compactness property of  $\mathfrak{G}_{[0,1],A}$  is established for the case that  $A' = \{0\}$  and  $\mathcal{L}$  is a countable first-order language.

**Theorem 3.13.** *Let  $A' = \{0\}$ . Every strongly consistent  $\mathcal{L}$ -theory in  $\mathfrak{G}_{[0,1],A}$  is contained in a maximally strongly consistent deductively Henkin  $\mathcal{L}'$ -theory such that  $\mathcal{L} \subseteq \mathcal{L}'$ .*

*Proof.* Let  $T$  be a strongly consistent  $\mathcal{L}$ -theory.  $T'$  will be constructed in countably many phases. Indeed,  $T'$  is a maximally strongly consistent theory containing the union of countably many maximally strongly consistent  $\mathcal{L}_i$ -theories  $T_i$  in which for every  $i \geq 1$ ,  $\mathcal{L}_i$  have a witness constant for each unprovable sentence  $\forall x \varphi(x)$  where  $\varphi(x) \in \text{Form}(\mathcal{L}_{i-1})$ . To this end, consider the following notions.

- $\mathcal{L}_0 = \mathcal{L}$ .
- $F_0 = \text{Form}(\mathcal{L}_0)$  and for  $i \geq 1$ ,  $F_i = \text{Form}(\mathcal{L}_i) \setminus \text{Form}(\mathcal{L}_{i-1})$ .
- For each  $i \geq 1$ ,  $\mathcal{L}_i = \mathcal{L}_{i-1} \cup \{c_{\varphi(x),r,s} : \varphi(x) \in F_{i-1}, r, s \in A \cup \{1\}, r > s\}$  where each  $c_{\varphi(x),r,s}$  is a new constant symbol.
- $T'_0 = T$ .
- For nullary connectives  $\bar{r}$  and  $\bar{s}$  and formula  $\varphi(x)$ ,  $\theta_{\varphi(x),r,s} = (\bar{r} \rightarrow \forall x \varphi(x)) \vee (\varphi(c_{\varphi(x),r,s}) \rightarrow \bar{s})$ .
- For each  $i \geq 1$ ,  $T'_i = T_{i-1} \cup \{\theta_{\varphi(x),r,s} : \varphi(x) \in F_{i-1}, r, s \in A \cup \{1\}, r > s\}$  where  $T_{i-1}$  is a maximally strongly consistent  $\mathcal{L}_{i-1}$ -theory containing  $T'_{i-1}$ .

Firstly, we show that for each  $i \geq 0$ ,  $T'_i$  is strongly consistent. Obviously,  $T'_0$  is strongly consistent. Assume that for each  $k < n$ ,  $T'_k$  is strongly consistent. Suppose that, on the contrary,  $T'_n$  is not strongly consistent. Thus, there exists  $t \in A \cup \{1\}$  such that  $T'_n \vdash \bar{t}$ . Hence, there is a finite subset  $S$  of  $T_{n-1}$  such that  $S \cup \{\theta_{\varphi_i(x),r_i,s_i}\}_{i=1}^m \vdash \bar{t}$  and no proper subset of  $S \cup \{\theta_{\varphi_i(x),r_i,s_i}\}_{i=1}^m$  proves  $\bar{t}$ . Set,  $\Gamma = \{\theta_{\varphi_i(x),r_i,s_i}\}_{i=1}^{m-1}$ . By deduction theorem,  $S \cup \Gamma \vdash \theta_{\varphi_m(x),r_m,s_m} \rightarrow \bar{t}$ . Consider the abbreviations  $\theta_m$  and  $c_m$  for  $\theta_{\varphi_m(x),r_m,s_m}$  and  $c_{\varphi_m(x),r_m,s_m}$ , respectively. Since  $\varphi(c_m) \rightarrow \bar{s}_m \vdash \theta_m$  we have  $S \cup \Gamma \vdash (\varphi(c_m) \rightarrow \bar{s}_m) \rightarrow \bar{t}$ , which leads to deduce that  $T_{n-1} \cup \Gamma \vdash \bar{s}_m \rightarrow \forall x \varphi_m(x)$ . On the other hand, as  $\bar{r}_m \rightarrow \forall x \varphi_m(x) \vdash \theta_m$  we have  $S \cup \Gamma \vdash (\bar{r}_m \rightarrow \forall x \varphi_m(x)) \rightarrow \bar{t}$  and so one could conclude that  $T_{n-1} \cup \Gamma \vdash \forall x \varphi_m(x) \rightarrow \bar{r}_m$ . Hence,  $T_{n-1} \cup \Gamma \vdash \bar{s}_m \rightarrow \bar{r}_m$  and so  $T_{n-1} \cup \Gamma \vdash \bar{t}$  which is a contradiction.

Secondly, let  $\mathcal{L}' = \bigcup_{n \geq 0} \mathcal{L}_n$  and take a maximally strongly consistent  $\mathcal{L}'$ -theory  $T'$ , containing  $\bigcup_{n \geq 0} T_n$ .  $T'$  is provably Henkin. Verily, if  $T' \not\vdash \forall x \varphi(x)$  for some  $\varphi(x) \in \text{Form}(\mathcal{L}')$  then by maximality of  $T'$  and Lemma 3.12-3 there is  $r \in A \cup \{1\}$  such that  $\bar{r} \rightarrow \forall x \varphi(x) \notin T'$ . Now, as  $A' = \{0\}$  take  $s \in A \cup \{1\}$  such that  $s < r$ . As,  $(\bar{r} \rightarrow \forall x \varphi(x)) \vee (\varphi(c_{\varphi(x),r,s}) \rightarrow \bar{s}) \in T'$ , maximality of  $T'$  and Lemma 3.12-2 implies that  $\varphi(c_{\varphi(x),r,s}) \rightarrow \bar{s} \in T'$ . Thus, by Lemma 3.12-3  $T' \not\vdash \varphi(c_{\varphi(x),r,s})$ , and the proof is completed.  $\square$

**Corollary 3.14.** *Let  $\mathcal{L}$  be a countable first-order language. If  $A' = \{0\}$  then  $\mathfrak{G}_{[0,1],A}$  satisfy the compactness property. Specially,  $\mathfrak{G}_{\downarrow}^*$  admits the compactness property.*

### 3.2. Entailment Compactness.

Now, we study the entailment compactness and approximate entailment compactness in Gödel logics  $\mathfrak{G}_{[0,1],A}$  and  $\mathfrak{G}_{[0,1],A}^{\Delta}$ . Note that the usual compactness follows from the entailment compactness. However, the method we use in this subsection based on the notion of "finitely entailment" while the

method used in the previous subsection is based on the notion of "proof" and the concept of "strongly consistency".

The following example show that the entailment compactness fails on  $\mathfrak{G}_\downarrow^*$ .

**Example 3.15.** let  $\mathcal{L} = \{\rho\}$  where  $\rho$  is a nullary predicate symbol. Let  $T = \{\frac{\overline{1}}{n} \rightarrow \rho\}_{n \in \mathbb{N}}$ . One can easily verify that in the Gödel logic  $\mathfrak{G}_\downarrow^*$ ,  $T \models \rho$  but  $T \not\models \rho$ .

However, when  $A' = \{0\}$ , the approximate entailment compactness holds in  $\mathfrak{G}_{[0,1],A}$ .

**Theorem 3.16.** *Let  $\mathcal{L}$  be a countable first-order language,  $T$  be an  $\mathcal{L}$ -theory, and  $\varphi$  be an  $\mathcal{L}$ -sentence. If  $A' = \{0\}$  then the Gödel logic  $\mathfrak{G}_{[0,1],A}$  enjoys the approximate entailment compactness. Particularly, in  $\mathfrak{G}_\downarrow^*$  we have,*

$$T \models \varphi \text{ if and only if } T \models^f \frac{\overline{1}}{n} \rightarrow \varphi \text{ for all integers } n \geq 1.$$

*Proof.* Let  $T \models \varphi$ . We want to show that  $T \models^f \bar{r} \rightarrow \varphi$  for all  $r \in A \cup \{1\}$ . Suppose not. So, there exists  $r_0 \in A \cup \{1\}$  such that for any finite subset  $S$  of  $T$ ,  $S \not\models \bar{r}_0 \rightarrow \varphi$ . Thus, for any finite subset  $S$  of  $T$  there is a model  $\mathcal{M}$  of  $S$  such that  $\mathcal{M} \not\models \bar{r}_0 \rightarrow \varphi$ , which means that  $\mathcal{M} \models \varphi \rightarrow \bar{r}_0$ . Thus, for any finite subset  $S$  of  $T$ ,  $S \cup \{\varphi \rightarrow \bar{r}_0\}$  is satisfiable. Hence, compactness property of  $\mathfrak{G}_{[0,1],A}$  implies that  $T \cup \{\varphi \rightarrow \bar{r}_0\}$  is satisfiable which is in contradict with  $T \models \varphi$ .

Conversely, suppose that for any  $r \in A \cup \{1\}$ ,  $T \models^f \bar{r} \rightarrow \varphi$ . We want to show that  $T \models \varphi$ . Since  $T$  has a model, by *reductio ad absurdum* suppose that there exists a model  $\mathcal{M}$  of  $T$  such that  $\mathcal{M} \not\models \varphi$ . But because  $A' = \{0\}$ , one could find  $r \in A \cup \{1\}$  such that  $\varphi^{\mathcal{M}} \geq r$ . This means for any finite subset  $S$  of  $T$ ,  $S \not\models \bar{r} \rightarrow \varphi$ , a contradiction.  $\square$

On the other hand,  $\mathfrak{G}_n^*$  enjoys the entailment compactness. This follows from the following two lemmas.

**Lemma 3.17.** *Let  $\mathcal{L}$  be a countable first-order language. Let  $A' = \emptyset$ ,  $T$  be a complete Henkin  $\mathcal{L}$ -theory, and  $\varphi$  be an  $\mathcal{L}$ -sentence. In the Gödel logic  $\mathfrak{G}_{[0,1],A}$ ,  $T \models \varphi$  if and only if  $T \models^f \varphi$ .*

*Proof.* Let  $\text{Lind}(T)$  be the class of all  $T$ -equivalent  $\mathcal{L}$ -sentences, i.e., the equivalence classes  $[\varphi]_T$  of all  $\mathcal{L}$ -sentences  $\varphi$  modulo to the following equivalence relation.

$$\varphi \sim \psi \text{ if and only if } T \models^f \varphi \leftrightarrow \psi.$$

By the same way as the proof of Theorem 3.8 and replacing  $\vdash$  by  $\models^f$  we obtain an  $\mathcal{L}$ -structure  $\mathcal{M}_T \models T$ .

Now, let  $T \models \varphi$  but  $T \not\models^f \varphi$ . So,  $[\varphi]_T \not\geq [\bar{0}]_T$ . But then since  $A' = \emptyset$ , the proof of Lemma 3.5 show that  $\varphi^{\mathcal{M}} = g([\varphi]_T) > 0$ , a contradiction.  $\square$

*Remark 3.18.* If  $A' = \{0\}$  and  $[\varphi]_T \geq [\bar{0}]_T$ , then  $g([\varphi]_T)$  does not necessarily greater than 0 (cf. proof of Lemma 3.5, case 2).

**Lemma 3.19.** *Let  $T$  be an  $\mathcal{L}$ -theory,  $\varphi$  be an  $\mathcal{L}$ -sentence, and  $T \not\models^f \varphi$ . The followings are hold in Gödel logics  $\mathfrak{G}_{[0,1],A}^\Delta$ .*

- (1) *There exists a complete  $\mathcal{L}$ -theory  $\bar{T} \supseteq T$  such that  $\bar{T} \not\models \varphi$ .*
- (2) *There exists a first-order language  $\mathcal{L}' \supseteq \mathcal{L}$  and a complete Henkin  $\mathcal{L}'$ -theory  $T' \supseteq T$  such that  $T' \not\models \varphi$ .*

*Proof.* Proof of (1) is straightforward. For (2) assume that

- $\mathcal{L}_0 = \mathcal{L}$ ,  $T_0 = T$ ,  $\epsilon_0 = \varphi$ ,
- for  $n \geq 0$ ,  $\bar{T}_n$  be a complete theory containing  $T_n$  such that  $\bar{T}_n \not\models \epsilon_n$ ,
- for  $n \geq 0$ ,  $\mathcal{L}_{n+1} = \mathcal{L}_n \cup \{\epsilon_{n+1}\} \cup \{c_\psi : \bar{T}_n \not\models \forall x \psi(x)\}$ , where  $\epsilon_{n+1}$  is a new nullary predicate symbol and each  $c_\psi$  is a new constant symbol,
- for  $n \geq 0$ ,  $T_{n+1} = \bar{T}_n \cup \{\epsilon_n \rightarrow \epsilon_{n+1}\} \cup \{\psi(c_\psi) \rightarrow \epsilon_{n+1} : \bar{T}_n \not\models \forall x \psi(x)\}$ .

We show that for each  $n \geq 0$ ,  $T_n \not\models \epsilon_n$ . Obviously  $T_0 \not\models \epsilon_0$ . Assume that  $T_n \not\models \epsilon_n$ . We show that  $T_{n+1} \not\models \epsilon_{n+1}$ . To this end, let  $A = B \cup \{\psi_i(c_{\psi_i}) \rightarrow \epsilon_{n+1} : \bar{T}_n \not\models \forall x \psi_i(x)\}_{i=1}^m$  be a finite subset of  $T_{n+1}$  in which  $B$  is a finite subset of  $\bar{T}_n$ . Since  $\bar{T}_n \not\models \epsilon_n$  and for each  $1 \leq i \leq m$ ,  $\bar{T}_n \not\models \forall x \psi_i(x)$  and  $\bar{T}_n$  is a complete  $\mathcal{L}_n$ -theory,  $\bar{T}_n \not\models \epsilon_n \vee (\bigvee_{1 \leq i \leq m} \forall x \psi_i(x))$ . Now, as  $B$  is a finite subset of  $\bar{T}_n$ , there is an  $\mathcal{L}_n$ -structure  $\mathcal{M} \models B$  such that  $\min\{\epsilon_n^{\mathcal{M}}, (\forall x \psi_1(x))^{\mathcal{M}}, \dots, (\forall x \psi_m(x))^{\mathcal{M}}\} = \alpha > 0$ . Interpreting  $\epsilon_{n+1}$  in  $\mathcal{M}$  by a nonzero rational number less than  $\alpha$  leads to the fact that  $\mathcal{M} \models A$  and  $\mathcal{M} \not\models \epsilon_{n+1}$ .

Now, let  $\mathcal{L}' = \bigcup_{n=0}^{\infty} \mathcal{L}_n$  and  $T^* = \bigcup_{n=0}^{\infty} T_n$ . One could easily verify that  $T^* \not\models \varphi$ . Let  $T'$  be a complete  $\mathcal{L}'$ -theory containing  $T^*$  such that  $T' \not\models \varphi$ . Obviously the construction implies that  $T'$  is a complete Henkin  $\mathcal{L}'$ -theory.  $\square$

Now, in the light of Lemma 3.17 and Lemma 3.19 we deduced the entailment compactness of  $\mathfrak{G}_n^*$ .

**Corollary 3.20.** *Let  $\mathcal{L}$  be a countable first-order language.  $\mathfrak{G}_n^*$  enjoys the entailment compactness property.*

*Remark 3.21.* Let  $\mathcal{L}$  be first-order language. When  $A' = \emptyset$  one could easily modify the proof of Lemma 3.5 and Lemma 3.17 to the case of Gödel logics equipped with the unary connective  $\Delta$ . So by Lemma 3.19,  $\mathfrak{G}_{[0,1],A}^{\Delta}$  admit the compactness as well as the entailment compactness.

Despite the compactness property of  $\mathfrak{G}_{[0,1],A}^{\Delta}$  for finite set  $A$ , this fails in  $\mathfrak{G}_{[0,1],A}^{\Delta}$  if  $A' = \{0\}$ .

**Example 3.22.** Let  $\mathcal{L}$  contain a nullary predicate symbol  $\rho$  and let  $T = \{\rho \rightarrow \frac{1}{n}\}_{n \in \mathbb{N}} \cup \{\neg(\Delta(\rho))\}$ . One can easily verify that in the Gödel logic  $\mathfrak{G}_{[0,1],(0,1)_{\downarrow}}^{\Delta}$ ,  $T$  is finitely satisfiable but it is not satisfiable.

#### 4. COMPACTNESS WHEN THE UNDERLYING LANGUAGE IS UNCOUNTABLE

As already mentioned, all the compactness results in Gödel logics are restricted by the countability of the underlying language. The following example shows that when the underlying language is uncountable, the compactness fails in almost all extensions of Gödel logics.

**Example 4.1.** [Jeř14] Let  $\mathcal{L}$  be a relational language contains uncountably many unary predicate symbols  $\{R(x)\} \cup \{\rho_i(x)\}_{i \in (\omega_1+1)}$ . Set,

$$T = \{\neg\forall x R(x), \forall x (\rho_1(x) \Rightarrow R(x))\} \cup \{\forall x (\rho_i(x) \Rightarrow \rho_j(x)) : i > j\}_{i,j \in (\omega_1+1)}$$

Clearly, in any Gödel logic  $\mathfrak{G}_{[0,1],A}$ ,  $T$  is finitely satisfiable but it is not satisfiable.

However, when  $V' \subseteq \{0\}$  we show that the Gödel logic  $\mathfrak{G}_{V,A}$  may admit the compactness property, even for uncountable first order languages. Note that in the above example,  $T$  is not finitely satisfiable in Gödel logics with truth value set  $V$  such that  $V' \subseteq \{0\}$ .

We prove the compactness theorem by the ultraproduct method. For doing this we need a similarity relation. In metrically semantic of t-norm based fuzzy logics, the interpretation of similarity relation is a pseudo-metric.

#### 4.1. Ultrametric structure.

One of the most important tools in classical first-order logic is the equality relation. The advantage of this relation appears almost in all results of model theory of classical first-order logic.

Equality relation has some common properties, called *Similarity Axioms*.

- (1) (Reflexivity)  $\forall x (x \approx x)$ .
- (2) (Symmetry)  $\forall x \forall y (x \approx y \rightarrow y \approx x)$ .
- (3) (Transitivity)  $\forall x \forall y \forall z ((x \approx y \wedge y \approx z) \rightarrow x \approx z)$ .

Let  $d$  be a binary predicate symbol in the language  $\mathcal{L}_d$  that we want to role as equality relation in classical first-order logic. If  $\mathcal{M}$  be an  $\mathcal{L}_d$ -structure such that

$$\mathcal{M} \models \{\forall x d(x, x), \forall x \forall y (d(x, y) \rightarrow d(y, x)), \forall x \forall y \forall z ((d(x, y) \wedge d(y, z)) \rightarrow d(x, z))\},$$

then it is easy to see that  $d^{\mathcal{M}}$  must be a pseudo-ultrametric on the universe of  $\mathcal{M}$  (a pseudo-metric that satisfies the strong triangle inequality, that is  $d^{\mathcal{M}}(a, b) \leq \max\{d^{\mathcal{M}}(a, c), d^{\mathcal{M}}(b, c)\}$  for all  $a, b, c \in M$ ).

**Definition 4.2.** For a given language  $\mathcal{L}_d$ , containing a binary predicate symbol  $d$ , an  $\mathcal{L}_d$ -ultrametric structure or simply an *ultrametric structure* is an  $\mathcal{L}_d$ -structure  $\mathcal{M}$  where  $(M, d^{\mathcal{M}})$  is an ultrametric space.

**Example 4.3.** Any first-order structure with the discrete metric is an ultrametric structure.

**Example 4.4.** Let  $(M, d)$  be an ultrametric space. Define  $d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$ .  $(M, d')$  is an ultrametric structure.

In classical first-order logic when  $P$  is an n-ary predicate symbol and  $f$  is an n-ary function symbol and  $\bar{a} \in M^n$  is "equal" to  $\bar{b} \in M^n$ , we have  $P^{\mathcal{M}}(\bar{a}) = P^{\mathcal{M}}(\bar{b})$  (as subsets of  $M^n$ ) and also  $f^{\mathcal{M}}(\bar{a})$  is "equal" to  $f^{\mathcal{M}}(\bar{b})$ . These properties express the extensional identicality of  $\bar{a}$  and  $\bar{b}$  with respect to the equality relation, which is called *Congruence Axioms*.

- (4) For each n-ary predicate symbol  $P$ ,  
 $\forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n (\bigwedge_{i=1}^n (x_i \approx y_i) \rightarrow (P(x_1, \dots, x_n) \leftrightarrow P(y_1, \dots, y_n)))$ .
- (5) For each n-ary function symbol  $f$ ,  
 $\forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n (\bigwedge_{i=1}^n (x_i \approx y_i) \rightarrow (f(x_1, \dots, x_n) \approx f(y_1, \dots, y_n)))$ .

Enforcing models to satisfy the congruence axioms leads to the right definition of structures in first-order logic (that is the interpretation of an n-ary function symbol in a model  $\mathcal{M}$  would be a function  $f^{\mathcal{M}} : M^n \rightarrow M$  and the interpretation of an n-ary predicate symbol  $P$  would be a subset of  $M^n$ ).

Now, let the  $\mathcal{L}_d$ -structure  $\mathcal{M}$  satisfy the congruence axioms for each function and predicate symbol, i.e.,

$$\mathcal{M} \models \{\forall \bar{x} \forall \bar{y} (d(\bar{x}, \bar{y}) \rightarrow d(f(\bar{x}), f(\bar{y}))), \forall \bar{x} \forall \bar{y} (d(\bar{x}, \bar{y}) \rightarrow (P(\bar{x}) \leftrightarrow P(\bar{y})))\}_{f, P \in \mathcal{L}_d},$$

in which  $d(\bar{x}, \bar{y})$  is an abbreviation for  $\bigwedge_{i=1}^n d(x_i, y_i)$ . One can easily verify that for each function symbol  $f$ ,  $f^{\mathcal{M}} : (M^n, d^{\mathcal{M}}) \rightarrow (M, d^{\mathcal{M}})$  would be a 1-Lipschitz continuous function, in which  $d^{\mathcal{M}}$  is a pseudo-ultrametric and  $d^{\mathcal{M}}(\bar{a}, \bar{b}) = \max_{1 \leq i \leq n} \{d^{\mathcal{M}}(a_i, b_i)\}$  for every  $\bar{a}, \bar{b} \in M^n$ . Furthermore, for each predicate symbol  $P$ ,  $P^{\mathcal{M}} : (M^n, d^{\mathcal{M}}) \rightarrow (V, d_{max})$  is a 1-Lipschitz continuous function.

**Definition 4.5.** For a given language  $\mathcal{L}_d$ , containing a binary predicate symbol  $d$ , a *Lipschitz  $\mathcal{L}_d$ -structure* or simply a *Lipschitz structure*  $\mathcal{M}$  in the Gödel logic  $\mathfrak{G}_V$  is a nonempty pseudo-ultrametric space  $(M, d^{\mathcal{M}})$  called the universe of  $\mathcal{M}$  together with:

- a) for any n-ary predicate symbol  $P$  of  $\mathcal{L}$ , a 1-Lipschitz continuous function  $P^{\mathcal{M}} : (M^n, d^{\mathcal{M}}) \rightarrow (V, d_{max})$ ,
- b) for any n-ary function symbol  $f$  of  $\mathcal{L}$ , a 1-Lipschitz continuous function  $f^{\mathcal{M}} : M^n \rightarrow M$ ,
- c) for any constant symbol  $c$  of  $\mathcal{L}$ , an element  $c^{\mathcal{M}}$  in the universe of  $\mathcal{M}$ .

The following lemma is used to prove the compactness property for the Gödel logic  $\mathfrak{G}_{\downarrow}$ . It is used in constructing a model for a theory with ultraproduct method.

**Lemma 4.6.** *Let  $\mathcal{M}$  be a Lipschitz  $\mathcal{L}_d$ -structure. In the Gödel logic  $\mathfrak{G}_{V,A}$  for every  $\mathcal{L}_d$ -formula  $\varphi(\bar{x})$  and  $\bar{a}, \bar{b} \subseteq M$ ,*

$$d_{max}(\varphi^{\mathcal{M}}(\bar{a}), \varphi^{\mathcal{M}}(\bar{b})) \leq d^{\mathcal{M}}(\bar{a}, \bar{b}).$$

*Proof.* Using the 1-Lipschitz continuity of the interpretation of function and predicate symbols and also 1-Lipschitz continuity of  $\rightarrow : (V^2, d_{max}) \rightarrow (V, d_{max})$ , the proof is straightforward.  $\square$

The 1-Lipschitz continuity of the interpretation of function and predicate symbols in Lipschitz structures leads to obtain a *canonical ultrametric structure*  $\mathcal{M}/d$  from a given Lipschitz structure  $\mathcal{M}$ . Verily, the underlying universe of  $\mathcal{M}/d$  is the set of equivalence classes of  $M$  modulo the equivalence relation  $d^{\mathcal{M}}$ . Furthermore, the interpretation of symbols of the language could be defined by:

- $c^{\mathcal{M}/d} = [c^{\mathcal{M}}]_d$ ,
- $f^{\mathcal{M}/d}([a_1]_d, \dots, [a_n]_d) = f^{\mathcal{M}}(a_1, \dots, a_n)$ ,
- $P^{\mathcal{M}/d}([a_1]_d, \dots, [a_n]_d) = P^{\mathcal{M}}(a_1, \dots, a_n)$ .

**Definition 4.7.** An  $\mathcal{L}_d$ -theory  $T$  is called a *metric-satisfiable* theory if there is an ultrametric structure  $\mathcal{M}$  which models  $T$ . Similarly, when  $T$  has a Lipschitz model  $\mathcal{M}$ , we call  $T$  a *Lipschitz-satisfiable* theory. The notions of *finitely metric-satisfiable* and *finitely Lipschitz-satisfiable* theories are similarly defined.

#### 4.2. Ultraproduct method and the compactness theorem.

Let  $\mathcal{L}$  be a first-order language (of any cardinality) and  $T$  be a finitely satisfiable  $\mathcal{L}$ -theory. Let  $I = \mathcal{P}_{fin}(T)$  be the collection of all finite subsets of  $T$ . For every  $\varphi \in T$  assume that  $\varphi_T = \{\Sigma : \varphi \in \Sigma \text{ and } \Sigma \in I\}$ . Obviously  $\{\varphi_T : \varphi \in T\}$  has the finite intersection property, and so it is contained in an ultrafilter  $\mathcal{D}$  on  $I$ .

Bellow we list some facts and notions about (ultra)filters on topological spaces.

- [Wil04] A filter  $\mathcal{F}$  on a topological space  $X$  is convergent to  $x \in X$  if for all open sets  $U$  containing  $x$ ,  $U \in \mathcal{F}$ .
- [Wil04]  $X$  is compact Hausdorff space if and only if every ultrafilter  $\mathcal{F}$  on  $X$  has a unique limit point.
- Let  $\{x_i\}_{i \in I}$  be a family of points of a topological space  $X$ . One could view  $\{x_i\}_{i \in I}$  as a function  $f : I \rightarrow X$ . If  $\mathcal{D}$  is an (ultra)filter on  $I$ , then  $I_{\mathcal{D}}(X) = \{A \subseteq X : f^{-1}(A) \in \mathcal{D}\}$  is an (ultra)filter on  $X$ . If  $I_{\mathcal{D}}(X)$  is convergent to an element  $x \in X$ , we call  $x$  a  $\mathcal{D}$ -limit of the family  $\{x_i\}_{i \in I}$ .
- Obviously  $x$  is a  $\mathcal{D}$ -limit of  $\{x_i\}_{i \in I}$  if and only if for each open set  $U$  containing  $x$ , the set  $\{i \in I : x_i \in U\}$  belongs to the (ultra)filter  $\mathcal{D}$ .



- If  $X$  is a compact Hausdorff and  $x$  is the unique  $\mathcal{D}$ -limit of the family  $\{x_i\}_{i \in I}$ , we write  $\lim_{\mathcal{D}} x_i = x$ .
- If  $X$  and  $Y$  are two compact Hausdorff topological spaces,  $f : X \rightarrow Y$  is a continuous function,  $\{x_i\}_{i \in I}$  is a family of elements of  $X$ , and  $\mathcal{D}$  is an ultrafilter on  $I$ , then  $f(\lim_{\mathcal{D}} x_i) = \lim_{\mathcal{D}} f(x_i)$ .

Now, we can prove the compactness theorem.

**Theorem 4.8.** *Let  $\mathcal{L}_d$  be a first-order language and  $V' \subseteq \{0\}$ . In the Gödel logic  $\mathfrak{G}_V$  as well as the Gödel logic  $\mathfrak{G}_{V, V \setminus \{0, 1\}}$ , every finitely Lipschitz-satisfiable theory  $T$  is Lipschitz-satisfiable.*

*Proof.* Let  $I = \mathcal{P}_{fin}(T)$  be the collection of all finite subsets of  $T$  and  $\mathcal{D}$  be an ultrafilter on  $I$  which contain  $\{\varphi_T : \varphi \in T\}$  where  $\varphi_T = \{\Sigma : \varphi \in \Sigma \text{ and } \Sigma \in I\}$ .

Since  $T$  is finitely Lipschitz-satisfiable, for each  $T_i \in I$  there is a Lipschitz structure  $\mathcal{M}_i$  which models  $T_i$ . Let  $M = \prod_{i \in I} M_i$  and for each  $n$ -ary predicate symbol  $R$  define  $R^{\mathcal{M}} : M^n \rightarrow V$  by

$$R^{\mathcal{M}}(\{x_{1i}\}_{i \in I}, \dots, \{x_{ni}\}_{i \in I}) = \lim_{\mathcal{D}} R^{\mathcal{M}_i}(x_{1i}, \dots, x_{ni}).$$

Note that  $V' \subseteq \{0\}$ . So  $(V, d_{max})$  is a compact Hausdorff space. Thus, the  $\mathcal{D}$ -limit of the family  $\{R^{\mathcal{M}_i}(x_{1i}, \dots, x_{ni})\}_{i \in I}$  is unique and therefore  $R^{\mathcal{M}}$  is well-defined.

One could easily verify that  $d^{\mathcal{M}}$  is a pseudo-ultrametric on  $M$ . Furthermore, Lipschitz continuity of  $\{R^{\mathcal{M}_i}\}_{i \in I}$  implies that  $R^{\mathcal{M}}$  is Lipschitz continuous.

Now, for each constant symbol  $c$  let  $c^{\mathcal{M}} = \{c^{\mathcal{M}_i}\}_{i \in I}$ . Also, for each  $n$ -ary function symbol  $f$  define  $f^{\mathcal{M}} : M^n \rightarrow M$  by  $f^{\mathcal{M}}(\{x_{1i}\}_{i \in I}, \dots, \{x_{ni}\}_{i \in I}) = \{f^{\mathcal{M}_i}(x_{1i}, \dots, x_{ni})\}_{i \in I}$ . Note that, by Lipschitz continuity of  $\{f^{\mathcal{M}_i}\}_{i \in I}$ ,  $f^{\mathcal{M}}$  is well-defined and Lipschitz continuous.

Finally, by an induction on the complexity of formulas and using the Lipschitz continuity of  $\{R^{\mathcal{M}_i}\}_{i \in I}$  and also Lipschitz continuity of logical connectives on  $(V, d_{max})$ , it is easy to see that for each formula  $\varphi(x_1, \dots, x_n)$  and elements  $a_k = \{a_{ki}\}_{i \in I}$  of  $M$  for  $1 \leq k \leq n$ ,

$$\varphi^{\mathcal{M}}(a_1, \dots, a_n) = \lim_{\mathcal{D}} \varphi^{\mathcal{M}_i}(a_{1i}, \dots, a_{ni}).$$

Thus,  $\mathcal{M}$  is a Lipschitz model of  $T$ . □

## 5. FINAL REMARKS AND FURTHER WORKS

We study the compactness property of extensions of first-order Gödel logic with truth constants or with the Baaz's  $\Delta$  connective. The compactness theorem is one of the basic theorems that is used in the model theory of classical first-order logic. Using the ultraproduct method, we prove the compactness theorem for theories whose underlying first-order language are uncountable. But, the ultraproduct method forced us to consider the Gödel sets with the reverse semantical meaning to interpret the similarity relation as a (pseudo) metric. Thus, semantically 0 is the absolute truth and 1 is the absolute falsity. The translation of results for the usual semantic of Gödel logic could be stated as follows:

- (1) (Corollary 3.10 and Corollary 3.20) If  $A' = \emptyset$  then for any countable first-order language  $\mathcal{L}$ , the Gödel logic  $\mathfrak{G}_{[0, 1], A}$  admits the compactness property as well as the entailment compactness.
- (2) (Corollary 3.14, Example 3.15, and Theorem 3.16) If  $A' = \{1\}$  then for any countable first-order language  $\mathcal{L}$ , the Gödel logic  $\mathfrak{G}_{[0, 1], A}$  admits the compactness property as well as the approximate entailment compactness, while the entailment compactness fails.
- (3) (Example 3.1) The compactness property fails in  $\mathfrak{G}_{[0, 1], A}$  whenever  $A$  has a limit point  $a \neq 1$  such that  $a \in A \cup \{0\}$ . In particular, RGL (first-order Gödel logic enriched with rational truth constants  $A = (0, 1) \cap \mathbb{Q}$  as nullary connectives) fails to have the compactness property.
- (4) (Theorem 4.8) If  $V' \subseteq \{1\}$  then in the Gödel logics  $\mathfrak{G}_V$  and  $\mathfrak{G}_{V, V \setminus \{0, 1\}}$  every finitely Lipschitz-satisfiable theory is Lipschitz-satisfiable.



- (5) (Remark 3.21) If  $A$  is finite then  $\mathfrak{G}_{[0,1],A}^\Delta$  admit the compactness property as well as the entailment compactness property.
- (6) (Example 3.1, Example 3.22) If  $A' = \{1\}$  or  $A' = \{0\}$  then the compactness fails in  $\mathfrak{G}_{[0,1],A'}^\Delta$ .

Regarding the first-order Gödel logic, it is seen that the absolute truth and absolute falsity has an asymmetry. Indeed, not falsity could be stated while it is impossible to separate truth from not truth. The outcome of equipping the Gödel logic with  $\Delta$  is objective as a kind of symmetry. Indeed, not only one could states "not truth" as well as the "not falsity", but also we have a symmetry in items (5) and (6), and also the results are hold in both semantical views of the Gödel logic.

A future interesting topic to study is the model theoretical aspects of  $\mathfrak{G}_\downarrow$ ,  $\mathfrak{G}_\downarrow^*$  and  $\mathfrak{G}_{[0,1]}^\Delta$ . Indeed, the expressive power of the language of this logics for stating " $\mathcal{M} \not\models \varphi$ " by means of " $\mathcal{M} \models \neg\Delta(\varphi)$ " or "there is a natural number  $n$  such that  $\mathcal{M} \models \varphi \rightarrow \frac{1}{n}$ ", helps us to develop the model theory of these logics.

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